On the Analysis of Population Protocols

Michael Blondin

Technical University of Munich

Overview

Population protocols: distributed computing model for massive networks of passively mobile finite-state agents

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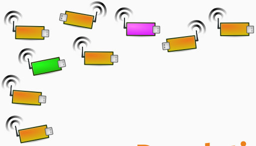

Population protocols: distributed computing model for massive networks of passively mobile finite-state agents

Can model *e.g.* networks of passively mobile sensors and chemical reaction networks

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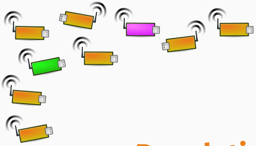

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Protocols compute predicates of the form $\varphi\colon \mathbb{N}^d \to \{0, 1\}$

e.g. if φ is unary, then $\varphi(n)$ is computed by n agents

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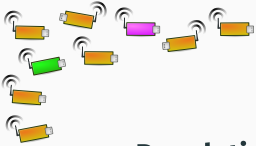

Population protocols: distributed computing model for massive networks of passively mobile finite-state agents

This talk:

- Automatic verification and testing
- Study of the minimal size of protocols

Population protocols

Angluin *et al.* PODC'04

- anonymous mobile agents with very few resources
- agents change states via random pairwise interactions
- each agent has opinion true/false
- computes by stabilizing agents to some opinion

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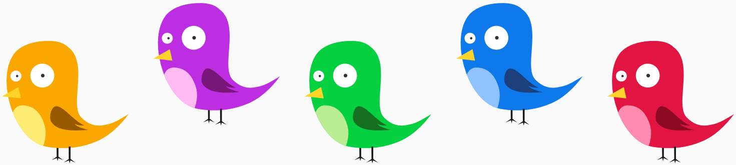

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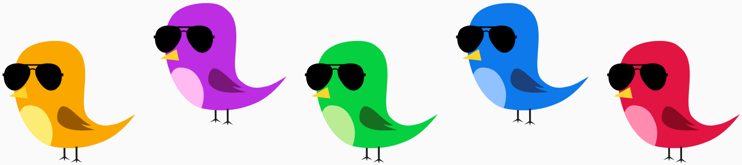

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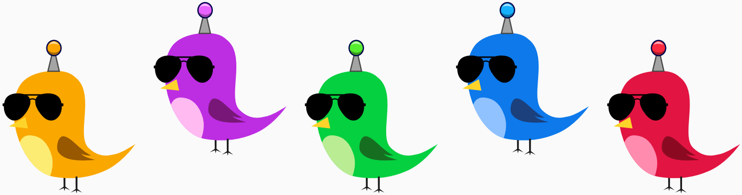

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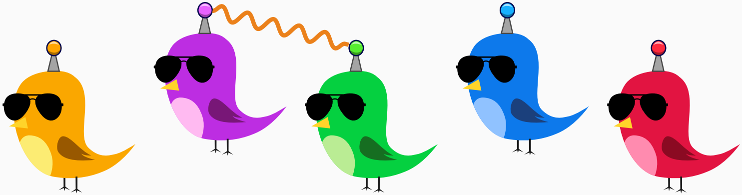

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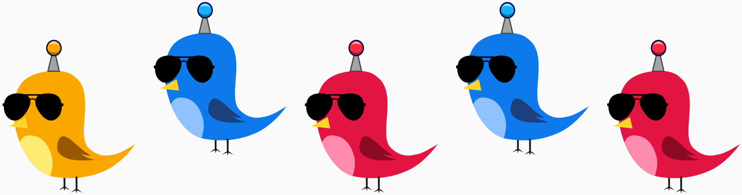

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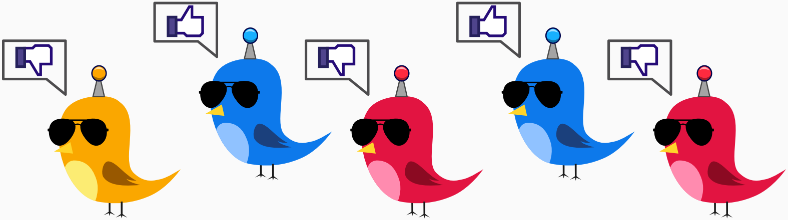

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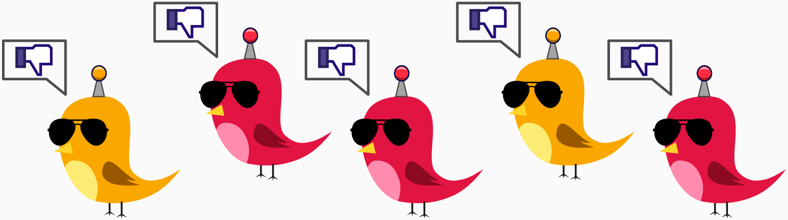

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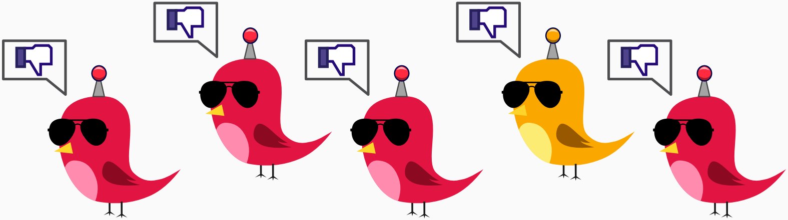

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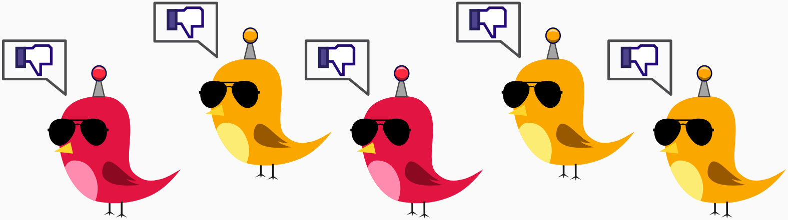
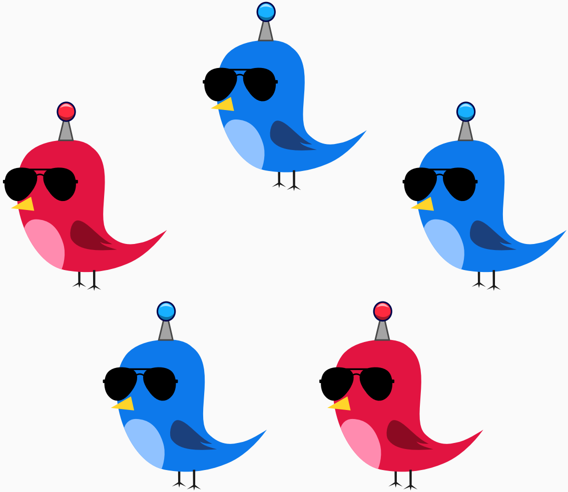


More blue birds than red birds?

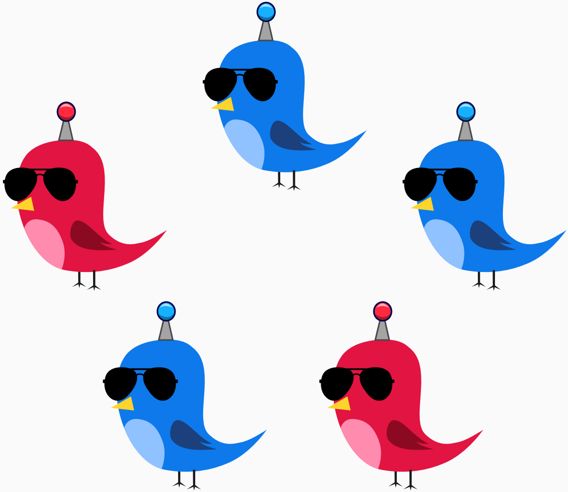


Example: majority protocol

More blue birds than red birds?

Protocol:

- Two large birds of different colors become small
- Large birds convert small birds to their color



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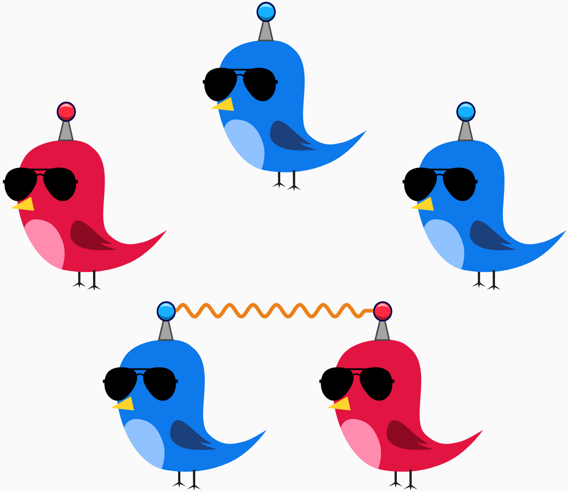

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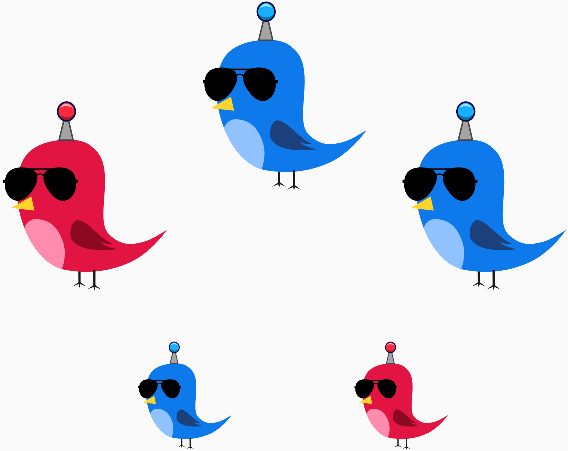

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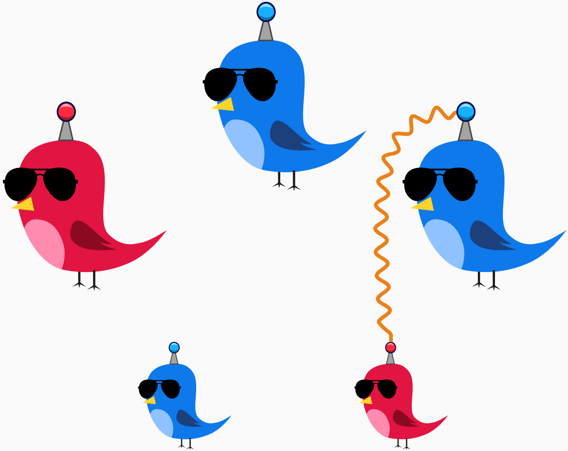

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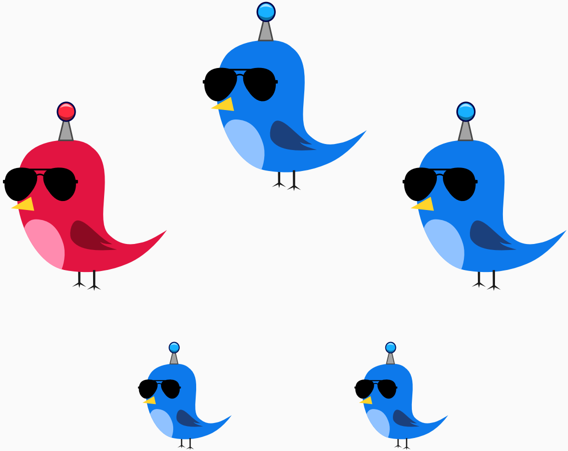

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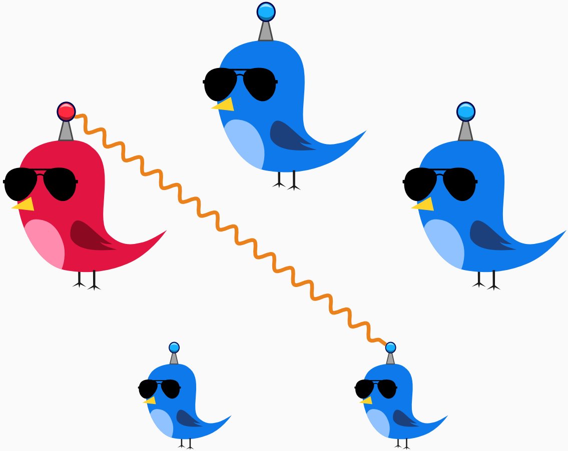

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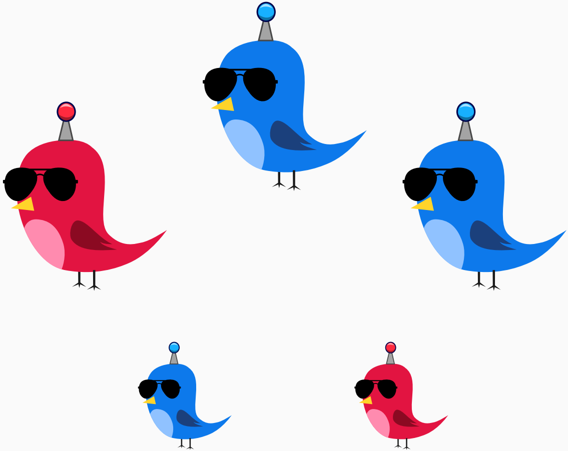

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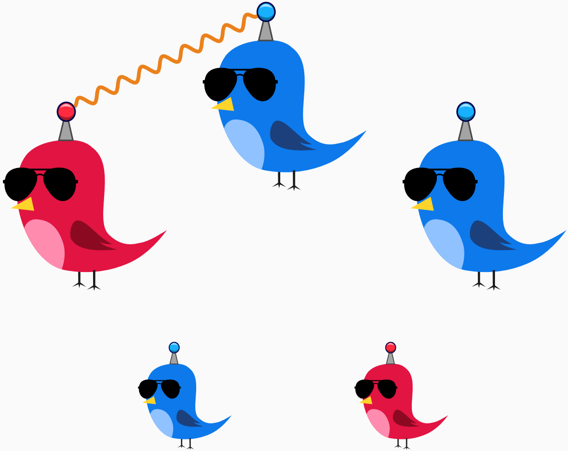

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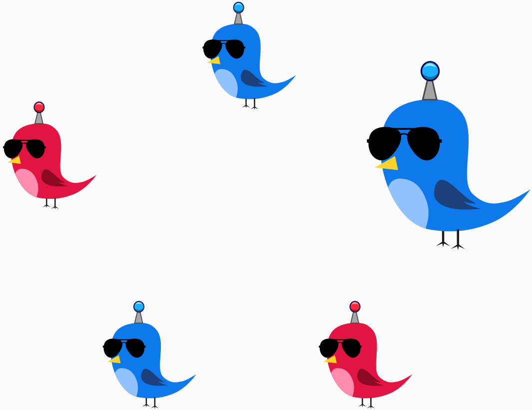

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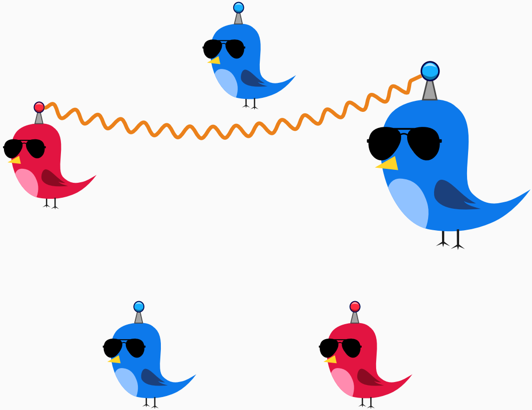

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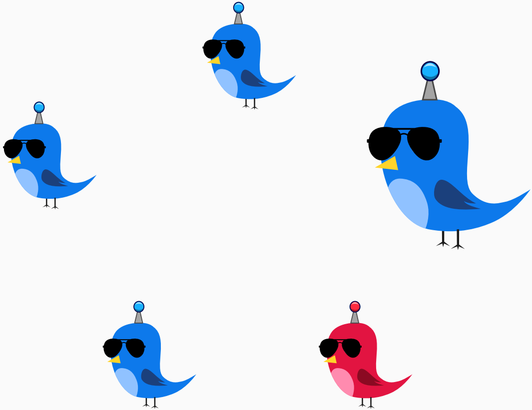

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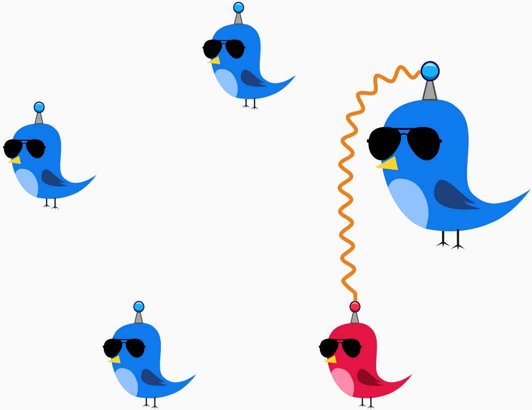

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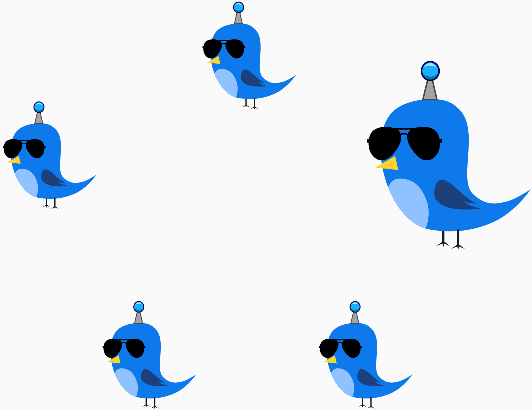

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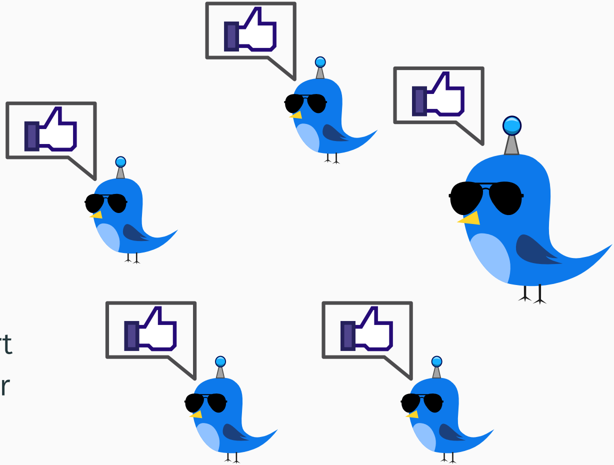

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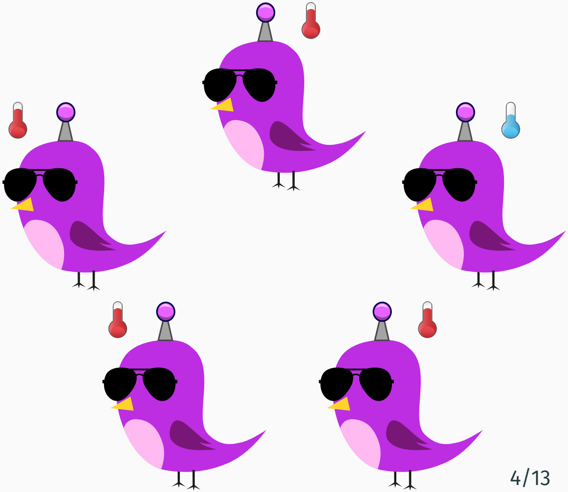
Are there at least 4 sick birds?



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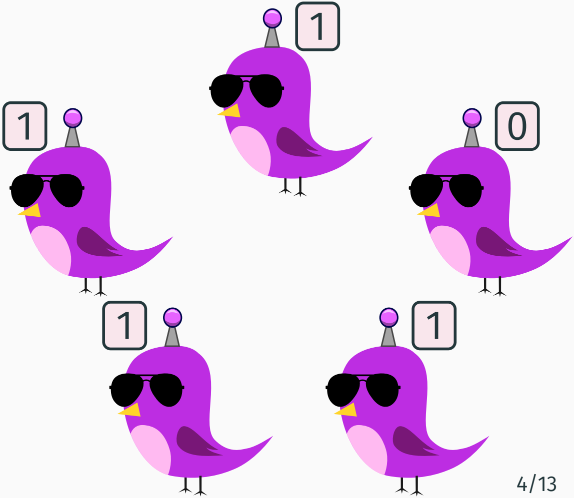

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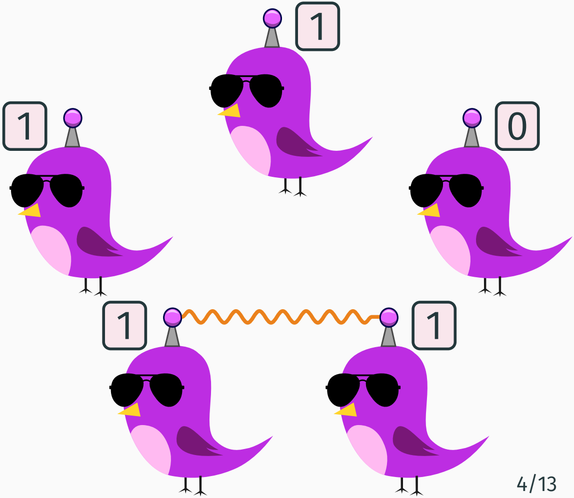

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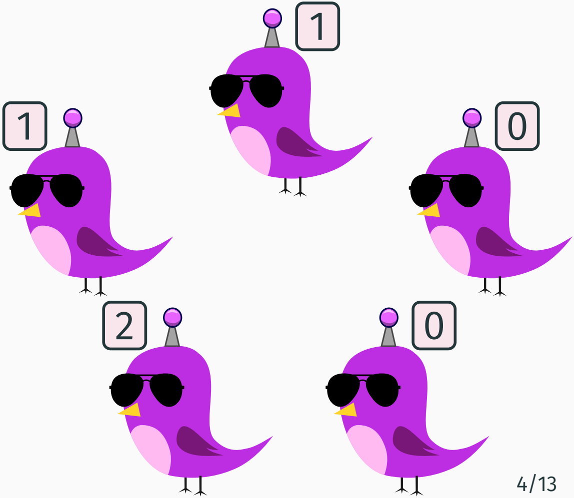

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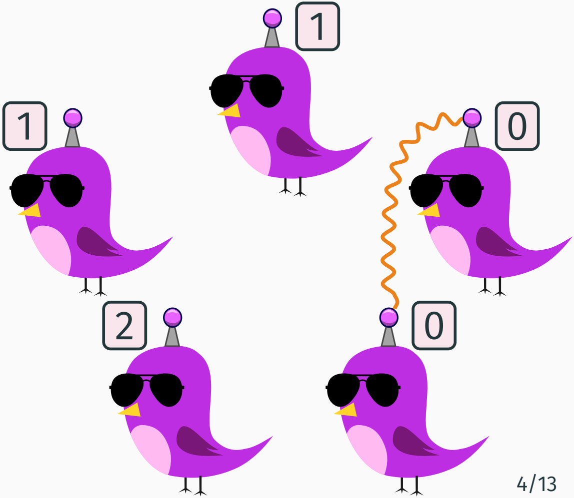

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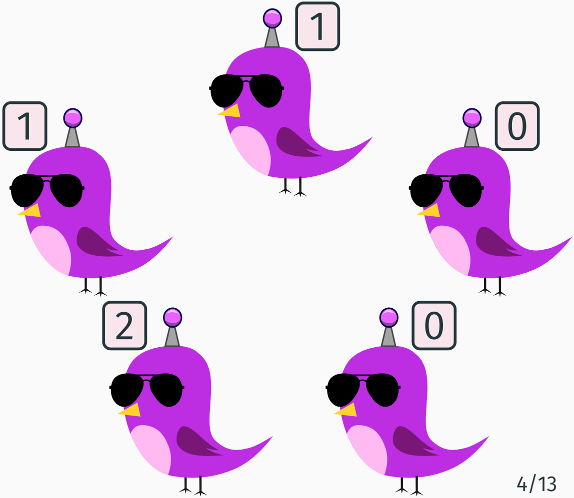

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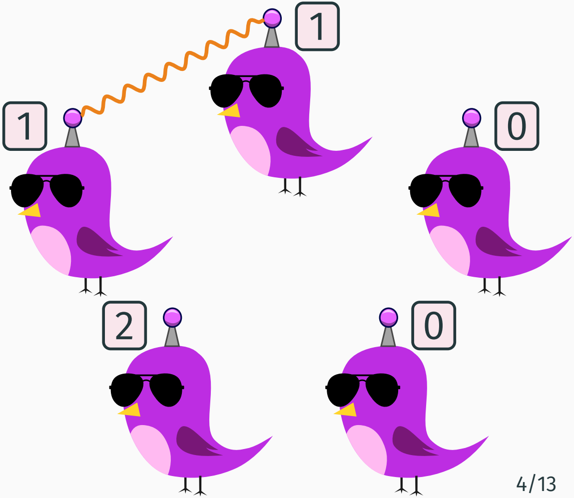

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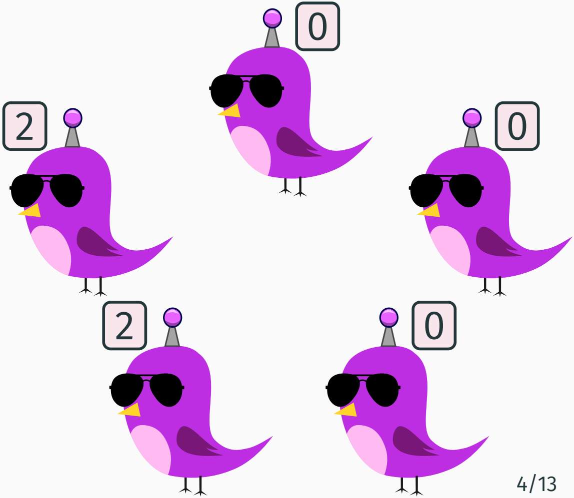

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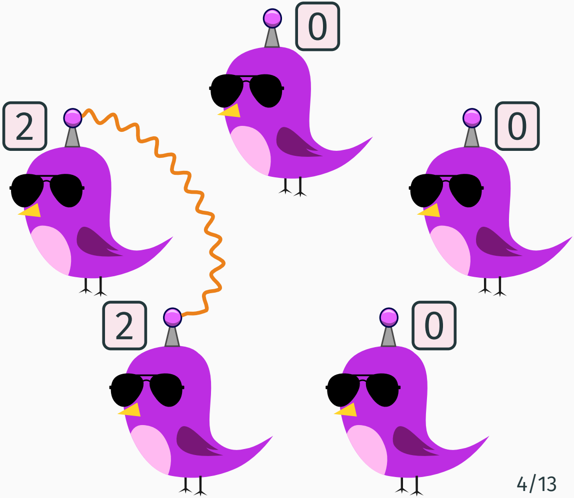

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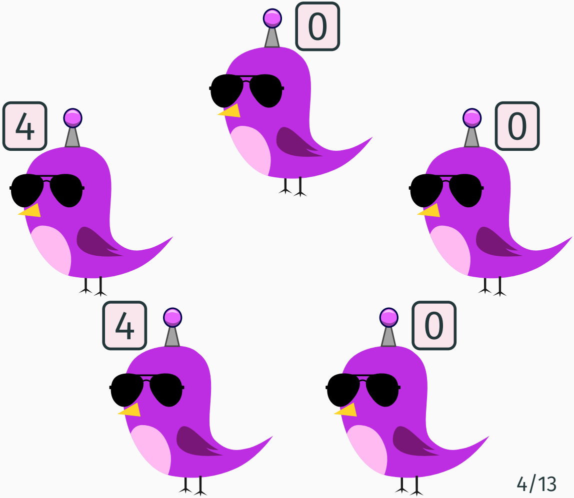

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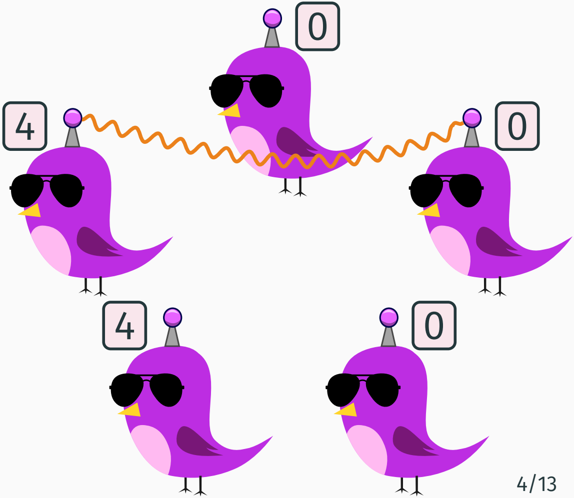

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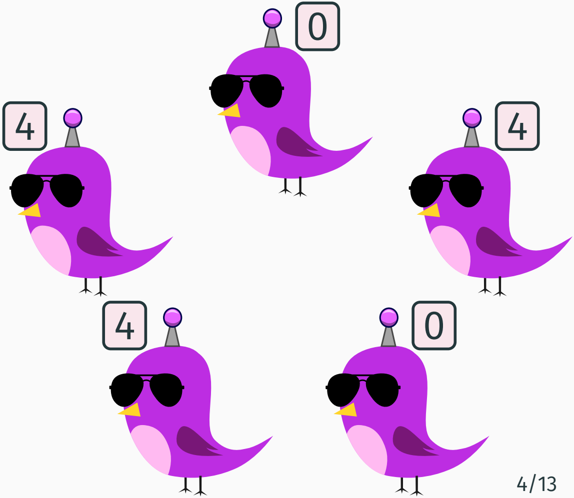

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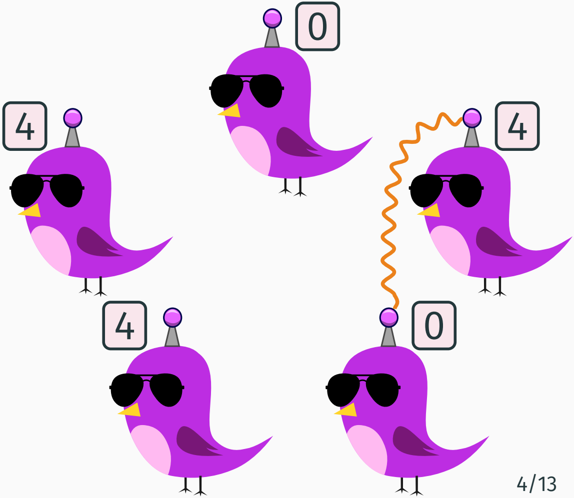

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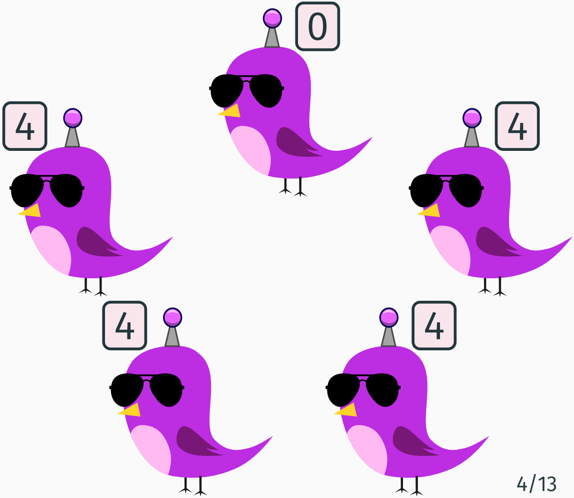

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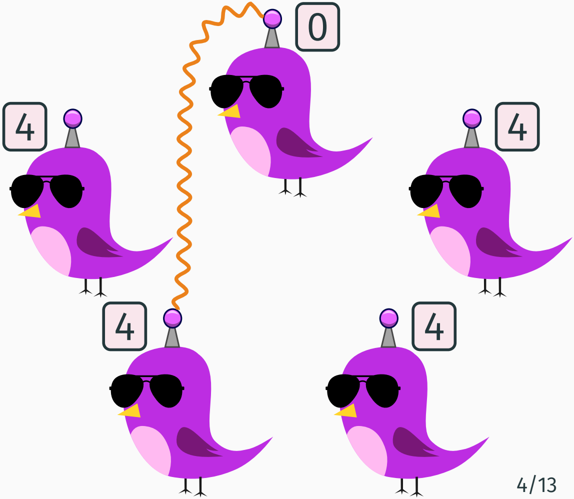

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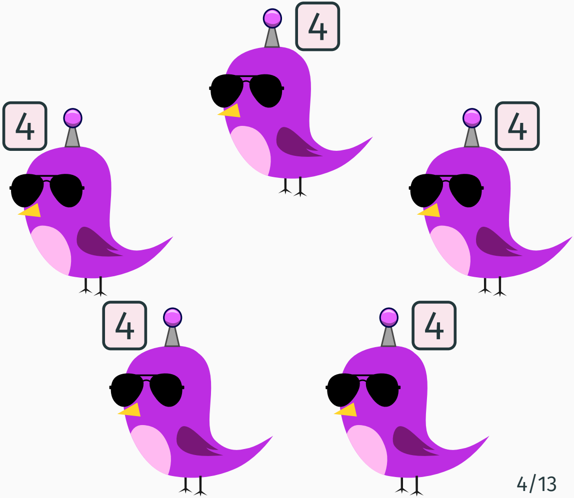

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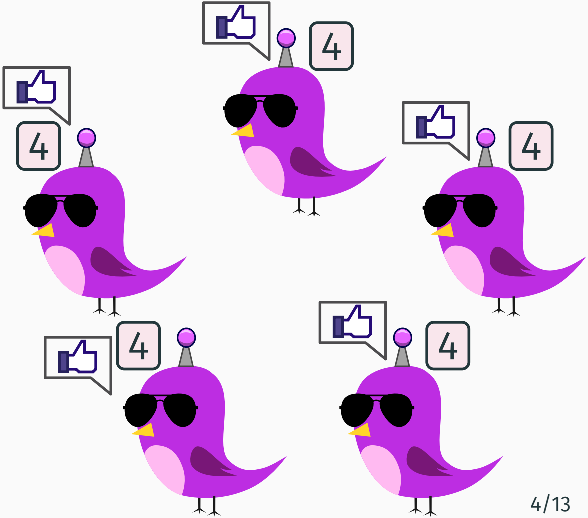
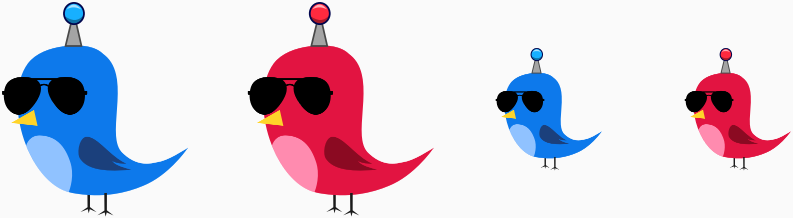


Demonstration

Population protocols: formal model

- *States*: finite set Q
- *Opinions*: $O : Q \to \{0, 1\}$
- *Initial states*: $I \subseteq Q$
- *Transitions*: $T \subseteq Q^2 \times Q^2$



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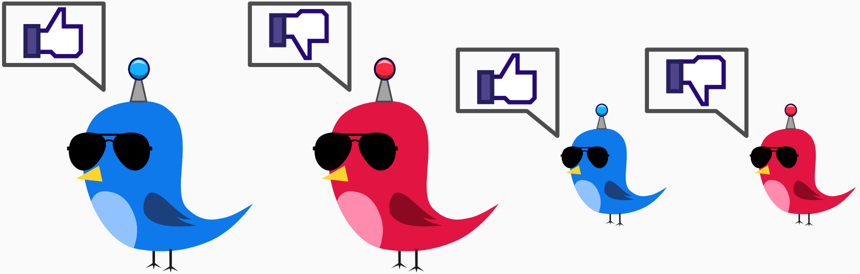

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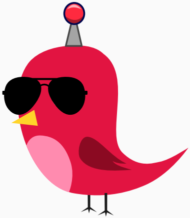

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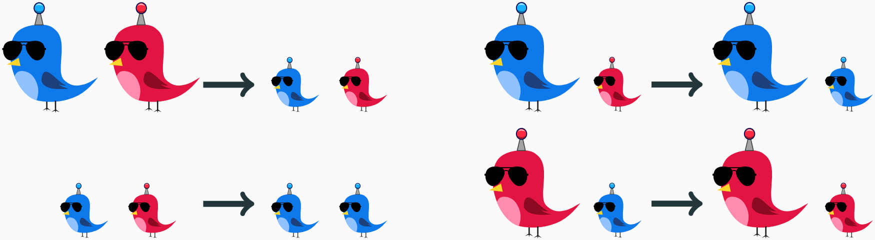
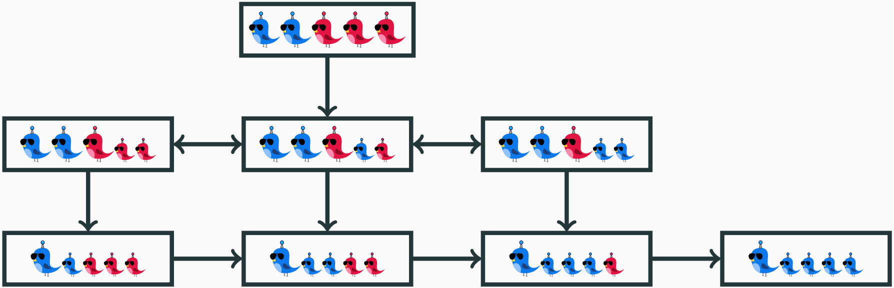
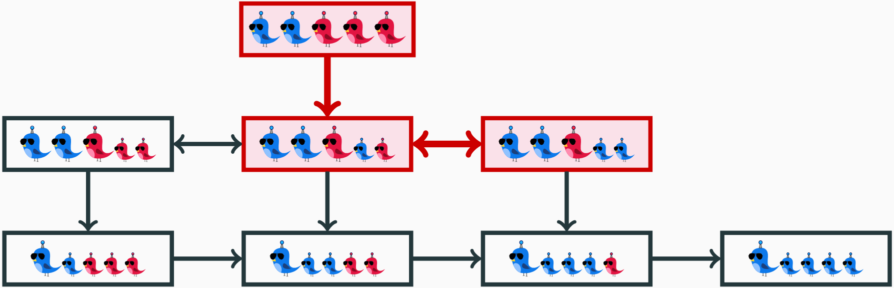
- *Initial states*: $I \subseteq Q$
- *Transitions*: $T \subseteq Q^2 \times Q^2$



Reachability graph:



Executions must be fair:



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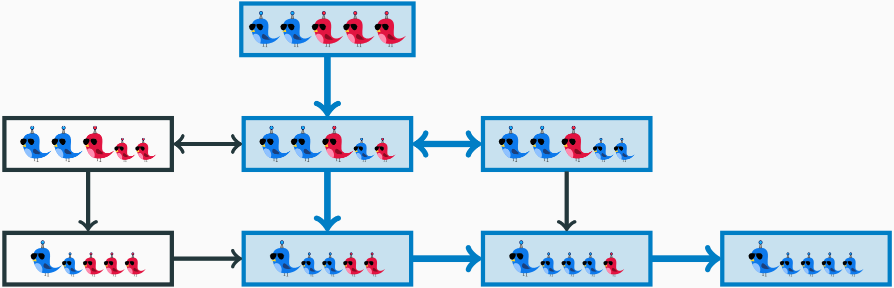
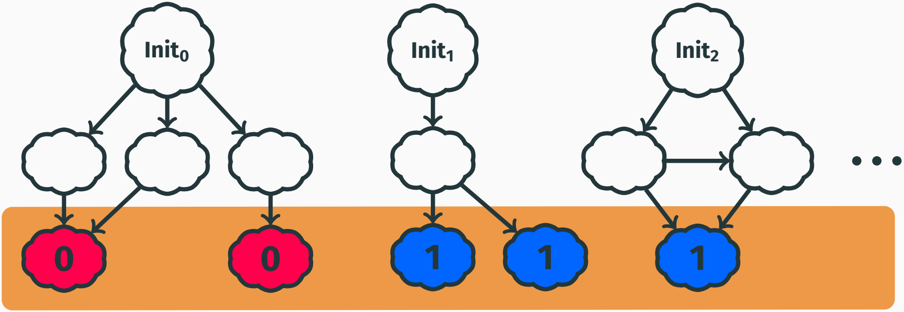


A protocol computes a predicate $f\colon \mathbb{N}^I \to \{0,1\}$

if fair executions reach common **consensus**



Population protocols: formal model

A protocol computes a predicate $f\colon \mathbb{N}^I \to \{0,1\}$

if fair executions reach common **consensus**

Expressive power **Angluin, Aspnes, Eisenstat PODC'06**

Population protocols compute precisely predicates definable in Presburger arithmetic, *i.e.* FO($\mathbb{N}$, $+$, $<$)

Protocols can become complex, even for B ≥ R:

Fast and Exact Majority in Population Protocols

Dan Alistarh
Microsoft Research

Rati Gelashvili*
MIT

Milan Vojnović
Microsoft Research

$weight(x) = \begin{cases} |x| & \text{if } x \in StrongStates \text{ or } x \in WeakStates; \\ 1 & \text{if } x \in IntermediateStates. \end{cases}$

$sgn(x) = \begin{cases} 1 & \text{if } x \in \{+0, 1_d, \ldots, 1_1, 3, 5, \ldots, m\}; \\ -1 & \text{otherwise.} \end{cases}$

$value(x) = sgn(x) \cdot weight(x)$

```
/* Functions for rounding state interactions */
```

$\phi(x) = -1_1$ if $x = -1; 1_1$ if $x = 1; x,$ otherwise

$R_\downarrow(k) = \phi(k$ if k odd integer, $k-1$ if k even$)$

$R_\uparrow(k) = \phi(k$ if k odd integer, $k+1$ if k even$)$

$Shift\text{-}to\text{-}Zero(x) = \begin{cases} -1_{j+1} & \text{if } x = -1_j \text{ for some index } j < d \\ 1_{j+1} & \text{if } x = 1_j \text{ for some index } j < d \\ x & \text{otherwise.} \end{cases}$

$Sign\text{-}to\text{-}Zero(x) = \begin{cases} +0 & \text{if } sgn(x) > 0 \\ -0 & \text{oherwise.} \end{cases}$

procedure update$\langle x, y \rangle$
  if $(weight(x) > 0$ *and* $weight(y) > 1)$ ***or*** $(weight(y) > 0$ *and* $weight(x) > 1)$ **then**
    $x' \leftarrow R_\downarrow\left(\frac{value(x)+value(y)}{2}\right)$ and $y' \leftarrow R_\uparrow\left(\frac{value(x)+value(y)}{2}\right)$
  else if $weight(x) \cdot weight(y) = 0$ *and* $value(x) + value(y) > 0$ **then**
    if $weight(x) \neq 0$ **then** $x' \leftarrow Shift\text{-}to\text{-}Zero(x)$ and $y' \leftarrow Sign\text{-}to\text{-}Zero(x)$
    else $y' \leftarrow Shift\text{-}to\text{-}Zero(y)$ and $x' \leftarrow Sign\text{-}to\text{-}Zero(y)$
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      $(y \in \{-1_d, +1_d\}$ *and* $weight(x) = 1$ *and* $sgn(y) \neq sgn(x))$ **then**
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How to verify correctness automatically?

Analysis of protocols

Number of states corresponds to amount of memory, relevant to keep it minimal for embedded systems

- **B ≥ R** requires at least 4 states (Mertzios *et al.* ICALP'14)
- **X ≥ c** requires at most c + 1 states

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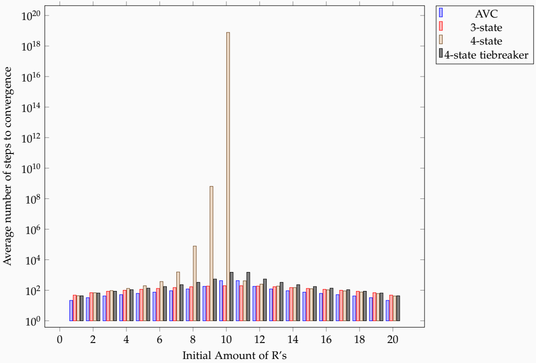
What is the state complexity of common predicates?

Convergence speed may vary wildly, challenging to establish bounds



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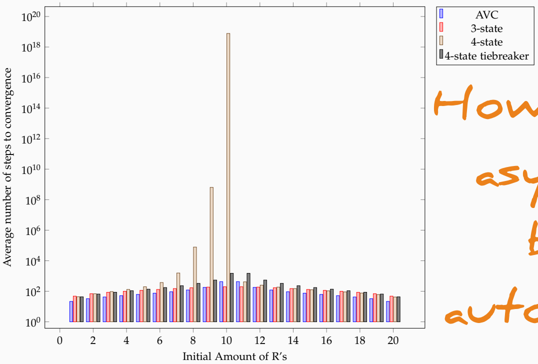

How to derive asymptotic bounds automatically?

1. **Automatic verification of correctness**
   - PODC'17 with Javier, Stefan and Philipp
   - Submission to CAV'18 with Javier and Stefan
   - Interns: Philip Offtermatt and Amrita Suresh
2. **State complexity of common predicates**
   - STACS'18 with Javier and Stefan
3. **Automatic analysis of convergence speed**
   - Ongoing work with Javier and Antonín Kučera

Analysis of protocols

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This talk

Existing verification tools:

- PAT: model checker with global fairness

  (Sun, Liu, Song Dong and Pang CAV'09)

- `bp-ver`: graph exploration

  (Chatzigiannakis, Michail and Spirakis SSS'10)

- Conversion to counter machines + `PRISM`/`Spin`

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Verification: state of the art

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Only for populations of fixed size!

Verification: state of the art

Sometimes possible to verify all sizes:

- Verification with the interactive theorem prover Coq

(Deng and Monin TASE'09)

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Not automatic!

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Challenge: verifying automatically all sizes

Testing whether a protocol computes φ amounts to testing:

$$\neg\exists C, D: \quad C \xrightarrow{*} D \wedge$$
$$C \text{ is initial } \wedge$$
$$D \text{ is in a BSCC} \wedge$$
$$\text{opinion}(D) \neq \varphi(C)$$

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As difficult as verification

Testing whether a protocol computes φ amounts to testing:

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Relaxed with Presburger-definable overapproximation!

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Difficult to express

Testing whether a protocol computes φ amounts to testing:

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BSCCs are of size 1 for most protocols!

Testing whether a protocol computes φ amounts to testing:

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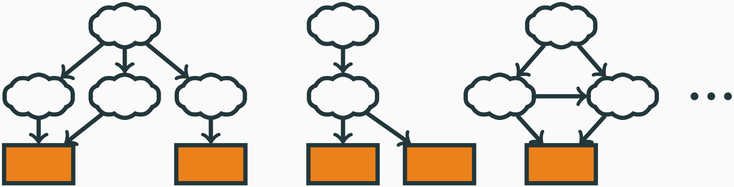
Testable with an SMT solver

Testing whether a protocol computes φ amounts to testing:

$$\begin{aligned}\neg\exists C, D: \quad & C \overset{*}{\dashrightarrow} D \wedge \\ & C \text{ is initial} \wedge \\ & D \text{ is terminal } \wedge \\ & \text{opinion}(D) \neq \varphi(C)\end{aligned}$$

But how to know whether all BSCCs are of size 1?

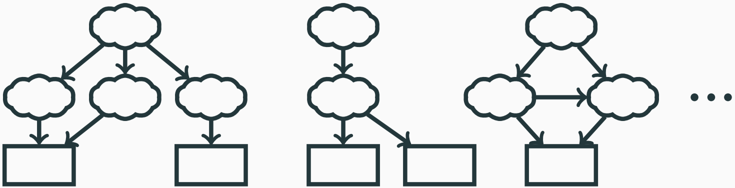
Protocol is *silent* if fair executions reach terminal configurations



BSCCs of size 1

Protocol is *silent* if fair executions reach terminal configurations

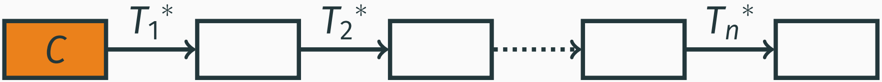
- Testing silentness is as hard as verification of correctness
- But most protocols satisfy a common design



BSCCs of size 1

Partition $T = T_1 \cup T_2 \cup \cdots \cup T_n$ s.t. for every i

- all executions restricted to T_i terminate
- if $T_1 \cup \cdots \cup T_{i-1}$ disabled in C and $C \xrightarrow{T_i{}^*} D$, then $T_1 \cup \cdots \cup T_{i-1}$ also disabled in D



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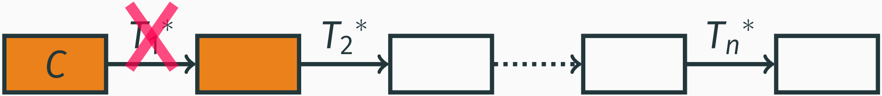

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Common design: layered termination

Partition $T = T_1 \cup T_2 \cup \cdots \cup T_n$ s.t. for every i

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T_1

$$\begin{aligned} B\ R &\rightarrow b\ r \\ R\ b &\rightarrow R\ r \\ B\ r &\rightarrow B\ b \\ b\ r &\rightarrow b\ b \end{aligned}$$

T_1

$$\mathbf{B}\ \mathbf{R} \to \mathbf{b}\ \mathbf{r}$$
$$\mathbf{R}\ \mathbf{b} \to \mathbf{R}\ \mathbf{r}$$
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Bad partition: not all executions over T_1 terminate

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$$\boxed{\begin{array}{l}\boldsymbol{R}\ \boldsymbol{b} \to \boldsymbol{R}\ \boldsymbol{r} \\ \boldsymbol{B}\ \boldsymbol{r} \to \boldsymbol{B}\ \boldsymbol{b}\end{array}}$$
$$\boldsymbol{b}\ \boldsymbol{r} \to \boldsymbol{b}\ \boldsymbol{b}$$

Bad partition: not all executions over T_1 terminate

$$\{\boldsymbol{B}, \boldsymbol{B}, \boldsymbol{R}, \boldsymbol{R}\} \to \{\boldsymbol{B}, \boldsymbol{b}, \boldsymbol{r}, \boldsymbol{R}\} \to \{\boldsymbol{B}, \boldsymbol{b}, \boldsymbol{b}, \boldsymbol{R}\} \to$$
$$\{\boldsymbol{B}, \boldsymbol{b}, \boldsymbol{r}, \boldsymbol{R}\} \to \{\boldsymbol{B}, \boldsymbol{b}, \boldsymbol{b}, \boldsymbol{R}\} \to \cdots$$

T_1

$$\mathbf{B}\ \mathbf{R} \to \mathbf{b}\ \mathbf{r}$$

T_2

$$\mathbf{R}\ \mathbf{b} \to \mathbf{R}\ \mathbf{r}$$

T_3

$$\mathbf{B}\ \mathbf{r} \to \mathbf{B}\ \mathbf{b}$$

$$\mathbf{b}\ \mathbf{r} \to \mathbf{b}\ \mathbf{b}$$

T_1	T_2	T_3
$B\ R \to b\ r$	$R\ b \to R\ r$	$B\ r \to B\ b$ $b\ r \to b\ b$

$\#B \geq \#R$:

$$\{B^*,\ R^*\}$$

T_1 ✗

$$B\,R \to b\,r$$

T_2

$$R\,b \to R\,r$$

T_3

$$B\,r \to B\,b$$
$$b\,r \to b\,b$$

$\#B \geq \#R$:

$$\{B^*,\ R^*\} \xrightarrow{*} \{B^*,\ b^*,\ r^*\}$$

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$\#B \geq \#R$:

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$\#R > \#B$:

$$\{R^+,\ B^*\}$$

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T_1 ✗

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$R\ b \to R\ r$

T_3

$B\ r \to B\ b$

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Theorem PODC'17

Deciding whether a protocol is strongly silent $\in$ NP

Proof sketch

Guess partition $T = T_1 \cup T_2 \cup \cdots \cup T_n$ and test whether it is correct by verifying

- Petri net structural termination
- Additional simple structural properties

Theorem PODC'17

Strongly silent protocols as expressive as general protocols

Proof sketch

- Protocols for

$$a_1x_1 + \ldots + a_nx_n \geq b$$

$$a_1x_1 + \ldots + a_nx_n \equiv b \pmod{m}$$

  have layered termination partitions

- Conjunction and negation preserve layered termination

Peregrine: Haskell + Z3 + JavaScript (front end)

gitlab.lrz.de/i7/peregrine

Protocol	Predicate	# states	# trans.	Time (secs.)
Majority [a]	$x \geq y$	4	4	0.1
Broadcast [b]	$x_1 \vee \cdots \vee x_n$	2	1	0.1
Linear ineq. [c]	$\sum a_i x_i \geq 9$	75	2148	2376
Modulo [c]	$\sum a_i x_i = 0 \bmod 70$	72	2555	3177
Threshold [d]	$x \geq 50$	51	1275	182
Threshold [b]	$x \geq 325$	326	649	3471
Threshold [e]	$x \geq 10^7$	37	155	19

[a] Draief *et al.* 2012
[b] Clément *et al.* 2011
[c] Angluin *et al.* 2006
[d] Chatzigiannakis *et al.* 2010
[e] Offtermatt 2017 (bachelor thesis)

Demonstration

Threshold state complexity: logarithmic bounds

Given: Presburger-definable predicate φ

Question: Smallest number of states necessary to compute φ?

Threshold state complexity: logarithmic bounds

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Difficult problem...
What about basic predicates?

Threshold state complexity: logarithmic bounds

Given: $c \in \mathbb{N}$

Question: Smallest number of states necessary to compute $x \geq c$?

Threshold state complexity: logarithmic bounds

Given: $c \in \mathbb{N}$

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Upper bound: $c + 1$

Lower bound: 2

Threshold state complexity: logarithmic bounds

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Theorem STACS'18

Computable with $O(\log c)$ states, if $c = 2^n$.

Proof sketch

$$
\begin{array}{ccc}
(1,1) & \mapsto & (2,0) \\
(2,2) & \mapsto & (4,0) \\
\vdots & & \vdots \\
(2^{n-1}, 2^{n-1}) & \mapsto & (2^n, 0) \\
(2^n, m) & \mapsto & (2^n, 2^n)
\end{array}
$$

Threshold state complexity: logarithmic bounds

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$$

+ extra states and transitions

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Theorem **STACS'18**

Let $P_0, P_1, \ldots$ be protocols such that P_c computes $x \geq c$.
There are infinitely many c s.t. P_c has $\geq (\log c)^{1/4}$ states.

Proof sketch

Counting argument on # unary predicates vs. # protocols.

Threshold state complexity: logarithmic bounds

Given: $c \in \mathbb{N}$

Question: Smallest number of states necessary to compute $x \geq c$?

Upper bound: $O(\log c)$

Lower bound: $\underbrace{O(\log^{1/4} c)}_{\text{for inf. many c}}$

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Possible to go below log c for some c?

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Lower bound: $\underbrace{O(\log^{1/4} c)}_{\text{for inf. many c}}$

Possible to go below log c for some c?

Yes!

Threshold state complexity: sublogarithmic bounds

Theorem — **STACS'18**

There exist protocols $P_0, P_1, \ldots$ and numbers $c_0 < c_1 < \cdots$ such that P_i computes $x \geq c_i$ and has $O(\log \log c_i)$ states.

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Lemma **Mayr and Meyer '82**

For every $c \in \mathbb{N}$, there exists a reversible multiset rewriting system $\mathcal{R}_c$ over alphabet $\Sigma \supseteq \{x, y, z, w\}$ of size $O(c)$ with rewriting rules $T \subseteq \Sigma^{\leq 5} \times \Sigma^{\leq 5}$ such that

$$\{x, y\} \xrightarrow{*} M \text{ and } w \in M \iff M = \{y, z^{2^{2^c}}, w\}$$

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Proof sketch

- $\mathcal{R}_c$ can be simulated by adding a padding symbol $\perp$

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Rewriting system $\mathcal{R}_c$	5-way population protocol
$(e, f, g) \mapsto (h, i)$	$(e, f, g, \perp, \perp) \mapsto (h, i, \perp, \perp, \perp)$
$(e, f) \mapsto (g, h, i)$	$(e, f, \perp, \perp, \perp) \mapsto (g, h, i, \perp, \perp)$

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Each 5-way transition is converted to a "gadget" of 2-way transitions

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- $\{w, w, \ldots, w\}$ reachable $\iff$ initially $\geq 2^{2^c}$ agents in $\perp$
- By reversibility and fairness, cannot avoid $\{w, w, \ldots, w\}$

State complexity: beyond threshold

Let $A \in \mathbb{Z}^{m \times k}$, let $\boldsymbol{c} \in \mathbb{Z}^m$ and let n be the largest absolute value of numbers occurring in A and $\boldsymbol{c}$.

Observation

Classical protocol computing $A\boldsymbol{x} + \boldsymbol{c} > \boldsymbol{0}$ has $O(n^m)$ states.

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Classical protocol computing $A\boldsymbol{x} + \boldsymbol{c} > \boldsymbol{0}$ has $O(n^m)$ states.

Theorem STACS'18

There exists a protocol that computes $A\boldsymbol{x} + \boldsymbol{c} > \boldsymbol{0}$ and has

- at most $O((m + k) \cdot \log mn)$ states
- at most $O(m \cdot \log mn)$ leaders

Conclusion

Peregrine:

- Graphical and command-line tool for designing, simulating and verifiying population protocols
- Can verify silent protocols

Future work:

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- Convergence speed analysis (ongoing with Javier and Tony)
- Failure ratio analysis
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- Complexity of $x \geq c$ can be decreased from $O(c)$ to $O(\log c)$ and sometimes $O(\log \log c)$
- Similar results for systems of linear inequalities

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  (not for the class of 1-aware protocols)
- State complexity of Presburger-definable predicates
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Thank you!   Vielen Dank!